TECHNISCHE UNIVERSITÄT BERLIN
Institut für Mathematik

Mathematical Physics I, Dr. Matteo Petrera, Dr. Raphael Boll WS 15/16
http://www3.math.tu-berlin.de/geometrie/Lehre/WS15/MP1/

Exercise Sheet 11 (20.01.16)

Due date: ***27.01.16***

- To get the Übungsschein (necessary condition for the oral exam) you need to collect *60% of the total sum of points*. Each exercise sheet has 30 points. The total number of sheets is 12-14.
- Please work in fixed groups of *2 students*.
- Please justify each step of your computations. *Results without any explanation are not accepted.* Please write in a readable way. *Unreadable handwriting will not be corrected.* Feel free to write your answers either in English or in German.
- Please turn in your homework directly to me at the beginning of the Tutorial or leave it in my letter box (MA 701, Frau Jean Downes). *No homework will be accepted after the deadline has passed.*

Exercise 1 (3 pts)

In the canonical phase space $\mathbb{R}^2$ consider the transformation

$$(q,p) \mapsto (\tilde{q},\tilde{p}) := \left(q\sqrt{1+q^2p^2}, \frac{p}{\sqrt{1+q^2p^2}} \right).$$

Show that this transformation is symplectic by proving that the canonical symplectic 2-form is preserved.

Exercise 2 (4+2+1+4+3 pts)

In the canonical phase space $\mathbb{R}^4$ consider the discrete dynamical system defined by iterations of the map $\Phi : \mathbb{R}^4 \setminus \{q_1^2 = q_2^2\} \to \mathbb{R}^4$ given by:

$$\begin{cases} \tilde{q}_1 = p_1(q_1^2+q_2^2) + 2q_1q_2p_2, \\ \tilde{q}_2 = p_2(q_1^2+q_2^2) + 2q_1q_2p_1, \\ \tilde{p}_1 = \dfrac{q_1}{q_1^2 - q_2^2}, \\ \tilde{p}_2 = -\dfrac{q_2}{q_1^2-q_2^2}. \end{cases}$$

TURN OVER!

1. Prove that the map Φ is symplectic.
2. Prove that the functions

$$F_1(q_1, q_2, p_1, p_2) := q_1 p_1 + q_2 p_2,$$
$$F_2(q_1, q_2, p_1, p_2) := q_1 p_2 + q_2 p_1,$$

are two functionally independent integrals of motion of Φ.
3. Prove that F_1 and F_2 are in involution, i.e., $\{F_1, F_2\} = 0$

Consider the change of coordinates $\Psi : (q_1, q_2, p_1, p_2) \mapsto (Q_1, Q_2, P_1, P_2)$ defined by:

$$\begin{cases} Q_1 = \dfrac{1}{2}\log|q_1^2 - q_2^2|, \\ Q_2 = \dfrac{1}{2}\log\left|\dfrac{q_1 + q_2}{q_1 - q_2}\right|, \\ P_1 = q_1 p_1 + q_2 p_2, \\ P_2 = q_1 p_2 + q_2 p_1. \end{cases}$$

4. Prove that the map Ψ is symplectic.
5. Prove that Ψ allows to *linearize* the map Φ, i.e., $\Psi \circ \Phi : \mathbb{R}^4 \setminus \{q_1^2 = q_2^2\} \to \mathbb{R}^4$ is given by

$$\begin{cases} \tilde{Q}_1 = Q_1 + \nu_1(F_1, F_2), \\ \tilde{Q}_2 = Q_2 + \nu_2(F_1, F_2), \\ \tilde{P}_1 = P_1, \\ \tilde{P}_2 = P_2. \end{cases}$$

Here ν_1 and ν_2 are two functions of the integrals of motion to be determined.

Exercise 3 (6 pts)

Determine the trajectory of a free (point-like and unit mass) particle moving on the surface of the sphere $\mathbb{S}^2$.

Exercise 4 (4+3 pts)

Consider two unit mass point-like particles in $\mathbb{R}^3$. The first particle moves along a circle, in a horizontal plane, parametrized by

$$(x_1, x_2, 0) = (r\cos\theta, r\sin\theta, 0), \qquad \theta \in \mathbb{R}/2\pi\mathbb{Z};$$

the second particle is constrained on the curve parametrized by

$$(y_1, y_2, y_3) = (r\cos\phi, r\sin\phi, h\sin\phi), \qquad \phi \in \mathbb{R}/2\pi\mathbb{Z},$$

where $r > 0$ and $h > 0$ are fixed. Assume that the two particles interact with an elastic force with constant $k > 0$. The gravitational field is neglected.

1. Write down the Lagrangian of the system.
2. Write down the Euler-Lagrange equations.